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Stability in Systems with Parameter

FRANK HOPPENSTEADT

Michigan State University, East Lansing, Michigan

Submitted by Norman Levinson

Suppose that for each set of values of the parameters $(a_1, \ldots, a_m)$ the system of real differential equations

$$\frac{dx_i}{dt} = f_i(x_1, \ldots, x_n, a_1, \ldots, a_m) \qquad i = 1, \ldots, n \tag{1}$$

has a rest point $x_1 = \varphi_1(a_1, \ldots, a_m), \ldots, x_n = \varphi_n(a_1, \ldots, a_m)$. If that rest point is asymptotically stable (Liapunov) for each set of values $(a_1, \ldots, a_m)$, in what sense, if any, is that stability uniform in the parameters $(a_1, \ldots, a_m)$?

Theorem 1 below gives an answer to this question, and a series of examples accompanying the theorem deals with some possible alternative results. Theorem 2 extends Theorem 1 to nonautonomous systems (i.e., systems whose right-hand sides depend explicitly on t).

A. N. Tihonov [1] applied a result of this type to the analysis of singular perturbation problems. The theorem used by Tihonov is incorrect, and because of this the proof of his convergence theorem for singular perturbation problems is inadequate. A proof of Tihonov's theorem is discussed below.

1. STATEMENT OF RESULTS

Let us write system (1) in vector notation,

$$x' = f(x, a) \qquad \left(' = \frac{d}{dt}\right). \tag{2}$$

It is assumed that f is continuous on some set $S_R \times F$ where

$$S_R = \left\{x \in E^n : |x| = \sum |x_i| \leqslant R\right\}$$

and F is some subset of E^m. Furthermore, we assume that $f(x, a)$ satisfies a

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Lipschitz condition for each $a \in F$ (i.e., for each $a \in F$ there exists a constant $M(a)$ such that $x, x^* \in S_R$ imply

$$| f(x, a) - f(x^*, a) | \leqslant M(| x - x^* |)).$$

Thus, the local existence and uniqueness of solutions of the initial value problem associated with (1) are assured. Moreover, solutions of (2) depend continuously on their initial conditions and the parameter a. Let $x = x(t, x_0, a)$ denote the solution of (2) which satisfies $x(0, x_0, a) = x_0$.

Suppose the equation

$$f(x, a) = 0$$

admits a continuous root $x = \varphi(a)$, $a \in F$. The transformation $y = x - \varphi(a)$ takes (2) into an equivalent system which has $y = 0$ as a rest point for each $a \in F$. In order to simplify the notation, let us assume that such a transformation has taken place; i.e., we assume

$$f(0, a) = 0 \qquad \text{for} \qquad a \in F.$$

The solution $x = 0$ of (2) is *stable uniformly in* $a \in F$ if for every $\epsilon > 0$ there is $\delta(\epsilon) > 0$ such that $| x_0 | < \delta$ implies $| x(t, x_0, a) | < \epsilon$ for $0 \leqslant t < \infty$ and $a \in F$. If there is a $\delta > 0$ such that $| x_0 | < \delta$ implies $| x(t, x_0, a) | \to 0$ as $t \to \infty$ for any $a \in F$, we say that the solution $x = 0$ of (2) has a *uniform domain of influence over* F. The solution $x = 0$ of (2) is *asymptotically stable uniformly in* $a \in F$ if it is stable uniformly in $a \in F$ and it has a uniform domain of influence over F.

The proof of the following theorem appears in the next section.

THEOREM 1. *Let F be a compact subset of E^m. If for each $a \in F$ the solution $x = 0$ of (2) is asymptotically stable, it is stable uniformly in $a \in F$.*

The examples below involve two-dimensional systems with scalar parameter ($x = (x_1, x_2)$, $a \in E^1$). For convenience these are written in polar coordinates

$$(r^2 = x_1{}^2 + x_2{}^2, \text{ and } \theta = \tan^{-1}(x_2/x_1)).$$

The conclusion drawn from the hypotheses of Theorem 1 by Tihonov is that the solution $x = 0$ of (2) is asymptotically stable uniformly in $a \in F$. The first example shows that this is not possible.

EXAMPLE 1.

$$r' = - r(r^2 - a^2)^2$$
$$\theta' = 1$$

with $F = [0, 1]$. For each a, $0 < a \leqslant 1$, only solutions which lie initially in the set $\{r : r < a\}$ tend to zero as $t \to \infty$. For $a = 0$ the asymptotic stability of $x_1 = 0$, $x_2 = 0$ is global. Clearly there can exist no $\delta > 0$ such that solutions beginning in $\{r : r < \delta\}$ for any $a \in F$ tend to zero as $t \to \infty$.

The next example shows that the compactness of F is needed in Theorem 1.

EXAMPLE 2.

$$\begin{aligned} r' &= r^2(r - a) \\ \theta' &= 1 \end{aligned}$$

with $F = (0, 1]$. For each $a \in F$ solutions of this system beginning in $\{r : r < a\}$ tend to zero as $t \to \infty$; the set $\{r : r = a\}$ is invariant; and, solutions beginning in $\{r : r > a\}$ are unbounded. The solution $x_1 = 0$, $x_2 = 0$ is not stable uniformly in $a \in F$ since for any $\delta > 0$ there are solutions of this system with $a < \delta$ which initially lie in $\{r : r < \delta\}$, but which are unbounded.

Finally, the conclusion of Theorem 1 need not hold if we require that the zero solution of (2) be stable rather than asymptotically stable for each $a \in F$.

EXAMPLE 3.

$$\begin{aligned} r' &= g(r, a) \\ \theta' &= 1 \end{aligned}$$

with $F = [0, 1]$ where

$$g(r, a) = \begin{cases} 0 & 0 \leqslant r \leqslant a \\ ar^5(r^2 - a^2) & a < r < \infty. \end{cases}$$

Given $\epsilon > 0$, for each a, $0 < a \leqslant 1$, solution of this system must initially lie in the set $\{r : r < \min(a, \epsilon)\}$ to ensure that it remains in $\{r : r < \epsilon\}$ for $0 \leqslant t < \infty$. For $a = 0$, any solution satisfying $r < \epsilon$ initially must remain in $\{r : r < \epsilon\}$ for $0 \leqslant t < \infty$. Thus, for each $a \in F$, the solution $x_1 = 0$, $x_2 = 0$ is stable. Clearly, that solution is not stable uniformly in $a \in F$.

In [4], Tihonov considers systems of the form

$$\begin{aligned} u' &= F(t, u, v) \qquad & u(t_0) &= u_0 \\ \epsilon v' &= G(t, u, v) \qquad & v(t_0) &= v_0 , \end{aligned}$$

where ϵ is a small positive parameter. Use is made of the auxiliary differential equation

$$\frac{dv}{ds} = G(t, u, v) \qquad v(0) = v_0 , \tag{3}$$

where s, $0 \leqslant s < \infty$, is a new independent variable and (t, u) are treated as parameters. Suppose $G(t, u, 0) = 0$ for all (t, u) in some compact set F. Under conditions not given here, Tihonov states: If the solution $v = 0$ of (3) is asymptotically stable for each $(t, u) \in F$, for small ϵ the solution of the perturbed system approximates the solution of

$$u' = F(t, u, v) \qquad u(t_0) = u_0$$
$$0 = G(t, u, v)$$

uniformly on compact t sets not including t_0.

The proof of this theorem given by Tihonov relies on the asymptotic stability of the zero solution of (3) being uniform in $(t, u) \in F$. As Example 1 shows, this does not follow from the hypotheses. Using Theorem 1 above we see that this condition is fulfilled if in addition to the zero solution of (3) being asymptotically stable for each $(t, u) \in F$ it has a uniform domain of influence over F. A simple proof of Tihonov's theorem for the case where the zero solution of (3) is asymptotically stable uniformly in $(t, u) \in F$ can be given through the use of Liapunov functions (see Lemma 4, [2]).

A sufficient condition for the existence of a uniform domain of influence over F for the zero solution of (2) is the following. Suppose $f(x, a)$ is twice continuously differentiable on $S_R \times F$. If the eigenvalues of the matrix $f_x(0, a)$ lie strictly in the left half of the complex plane for each $a \in F$, the solution $x = 0$ of (2) has a uniform domain of influence over F. A proof of this proceeds just as the proof of the Perron stability theorem ([3], p. 314).

Finally a result similar to Theorem 1 is possible for the nonautonomous case,

$$x' = f(t, x, a). \tag{3}$$

Since asymptotic stability in the autonomous case corresponds to uniform-asymptotic stability in the nonautonomous case (Massera [4]), the hypotheses must be changed accordingly.

THEOREM 2. *Suppose F is a compact subset of E^m and $f(t, x, a)$ is uniformly continuous on $[0, \infty) \times S_R \times F$. If $x = 0$ is a uniform-asymptotically stable solution of (3) for each $a \in F$, it is stable uniformly in $a \in F$.*

Examples are easily constructed which show that the uniform continuity of f and the uniform-asymptotic stability of the zero solution of (3) for each $a \in F$ are needed for this result. We present below the proof for the autonomous case. The proof for the nonautonomous case proceeds in exactly the same way.

2. PROOF OF THEOREM 1

According to Massera [4], for each $a \in F$ there exists a function $W(x; a)$ defined for $0 \leqslant |x| \leqslant \pi(a)$ for some $\pi(a) > 0$ with the following properties:

(i) $W(x; a)$, $W_x(x; a) \in C[S_{\pi(a)}]$;

(ii) $W(0; a) = 0$ and $W(x; a) > 0$ for $0 < | x | \leqslant \pi(a)$; and

(iii) $W_x(x; a) \cdot f(x, a) < 0$ for $0 < | x | \leqslant \pi(a)$.

Since $f(x, a)$ is continuous, for each $a_0 \in F$ there is an open set $U_{a_0} \subset S_{\pi(a_0)} \times F$ which contains $(S_{\pi(a_0)} - \{0\}) \times \{a_0\}$ such that

$$W_x(x; a_0) \cdot f(x, a) < 0 \qquad \text{for} \qquad (x, a) \in U_{a_0} .$$

Given $\epsilon > 0$ we shall find the desired $\delta(\epsilon) > 0$. Take $c > 0$ such that the set

$$\Gamma_c = \{x \in S_{\pi(a_0)} : W(x; a_0) \leqslant c\} \subset \operatorname{int} S_\epsilon .$$

Let

$$\Delta_c = \{x \in \Gamma_c : W(x; a_0) = c\},$$

and let

$$F_{a_0} = \{a \in F : \Delta_c \times \{a\} \subset U_{a_0}\} .$$

Since $a_0 \in F_{a_0}$ and since U_{a_0} is open, there exists $\eta(a_0) > 0$ such that $| a - a_0 | \leqslant \eta(a_0)$ implies $a \in F_{a_0}$.

Finally, there exists $\delta(\epsilon, a_0) > 0$ such that $| x_0 | < \delta(\epsilon, a_0)$ implies $| x(t, x_0, a) | < \epsilon$ for $0 \leqslant t < \infty$ and $| a - a_0 | \leqslant \eta(a_0)$. Indeed, take $\delta(\epsilon, a_0)$ such that $| x | < \delta(\epsilon, a_0)$ implies $W(x; a_0) < c$. If for some a^*, $| a^* - a_0 | \leqslant \eta(a_0)$ and some $| x^* | < \delta(\epsilon, a_0)$, $| x(T, x^*, a^*) | = \epsilon$ for some $T > 0$, let

$$\tau = \sup \{t < T : W(x(t, x^*, a^*); a_0) = c\}.$$

τ satisfies $0 < \tau < T$ since

$$W(x^*; a_0) < c < W(x(T, x^*, a^*); a_0).$$

Also, $W(x(\tau + p, x^*, a^*); \; a_0) > c$ for every p, $0 < p < T - \tau$. On the other hand, there exists p_0, $0 < p_0 < T - \tau$, such that

$$(x(t, x^*, a^*), a^*) \in U_{a_0} \qquad \text{for} \qquad \tau \leqslant t \leqslant \tau + p_0 .$$

Thus,

$$\begin{aligned} c &< W(x(\tau + p_0 , x^*, a^*); a_0) \\ &= W(x(\tau, x^*, a^*); a_0) + \int_\tau^{\tau+p_0} W_x(x(s, x^*, a^*); a_0) \cdot f(x(s, x^*, a^*), a^*) \, ds \\ &< W(x(\tau, x^*, a^*); a_0) = c. \end{aligned}$$

This contradiction establishes that such a point (x^*, a^*) cannot exist.

The set F is compact and contained in $\bigcup_{a \in F} \{b \in F : | b - a | < \eta(a)/2\}$. From this open covering of F we can extract a finite subcovering. Let $a^1, ..., a^N \in F$ be such that

$$F \subset \bigcup_{i=1}^{N} \left\{ b \in F : | b - a^i | < \frac{\eta(a^i)}{2} \right\}.$$

Let $\delta(\epsilon) = \min \{\delta(\epsilon, a^1), ..., \delta(\epsilon, a^N)\}$. Given $| x_0 | < \delta(\epsilon)$ and $a \in F$ there is a^j, $1 \leqslant j \leqslant N$, such that $| a - a^j | \leqslant \eta(a^j)/2$. Since $| x_0 | < \delta(\epsilon) \leqslant \delta(\epsilon, a^j)$, we have $| x(t, x_0, a) | < \epsilon$ for $0 \leqslant t < \infty$.

This completes the proof of Theorem 1.

References

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